PHILOSOPHICAL

TRANSACTIONS.

XVII. *Theorems for computing Logarithms. By the Rev.* John Hellins; *communicated by the Rev.* Nevil Maſkelyne, *D. D. F. R. S. and Aſtronomer Royal.*

Read March 16, 1780.

THE utility of logarithms is ſo well known, that much need not be ſaid upon it. In our days he muſt be a ſlender mathematician who does not know that they are uſeful, not only in trigonometry, navigation, aſtronomy, the calculations of compound intereſt

and annuities, but alſo in the finding of fluents, and the ſummation of infinite ſeries.

Some of the greateſt mathematicians that this kingdom ever produced, as Sir ISAAC NEWTON, Dr. HALLEY, Mr. COTES, and Mr. SIMPSON, have thought it not beneath them to improve the conſtruction of logarithms, which ſtrongly argues the utility of thoſe artificial numbers, and may ſuggeſt to us that the conſtruction of them cannot be much further improved.

Now, although we ſhould be very diffident in our expectations of improvement in any part of the mathematics after it has been handled by ſuch great men, yet, if the method of computing be in general long and tedious, or if there ſtill remain any particular difficulty, I believe, no good reaſon can be given why every attempt to abridge the one, or remove the other, ſhould be diſcouraged. The eaſy method of computing the logarithms of large numbers given in page 49. of Mr. SIMPSON's pamphlet on Trigonometry is a proof that thoſe gentlemen, who were of opinion that nothing better was to be hoped for, or expected, than what they publiſhed on the ſubject in the beginning of this century, were miſtaken. And the following theorems, inferior to none as to convergency, and uſeful in deducing the logarithms of great fractions from thoſe of ſmall ones, or the

the logarithms of ſmall numbers from thoſe of great ones, may be conſidered as another proof of the miſtake before mentioned. I have only to add here, that theſe theorems are new to me; and if they are ſo to the public, I humbly preſume they will be acceptable.

THEOREM I.

The log. of $\frac{p+q}{p} = 2 \times$ log. of $\frac{2p+2q}{2p+q}$ + log. of $\frac{\overline{2p+q}|^2}{\overline{2p+q}|^2 - qq}$.

DEMONSTRATION.

$$\frac{\overline{2p+q}|^2}{\overline{2p+q}|^2 - qq} \times \frac{\overline{2p+2q}|^2}{\overline{2p+q}|^2} = \frac{1}{\overline{2p+q}|^2 - qq} \times \frac{\overline{2p+2q}|^2}{1} = \frac{4pp+8pq+4qq}{4pp+4pq} =$$

$\frac{pp+2pq+qq}{p \times \overline{p+q}} = \frac{p+q}{p}$: conſequently, log. $\frac{\overline{2p+q}|^2}{\overline{2p+q}|^2 - qq} + 2$ l. $\frac{2p+2q}{2p+q}$ $=$ log. $\frac{p+q}{p}$. Q. E. D.

COROLLARY.

If $q=1$, and we write n for p, the theorem becomes log. $\frac{n+1}{n} = 2$ log. $\frac{2n+2}{2n+1}$ + log. $\frac{\overline{2n+1}|^2}{\overline{2n+1}|^2 - 1}$, which expreſſion perhaps is of more frequent uſe than that above.

THEOREM II.

Log. $\frac{p+q}{p} = 2$ log. $\frac{2p+q}{2p}$ − log. $\frac{\overline{2p+q}|^2}{\overline{2p+q}|^2 - qq}$.

DEMONSTRATION.

$\frac{2p+2q}{2p+q} \times \frac{2p+q}{2p} = \frac{2p+2q}{2p} = \frac{p+q}{p}$: therefore, log. $\frac{p+q}{p} =$ log. $\frac{2p+2q}{2p+q}$ + log. $\frac{2p+q}{2p}$. But it has been proved above, that log. $\frac{p+q}{p} = 2$ log. $\frac{2p+2q}{2p+q}$ + log. $\frac{\overline{2p+q}|^2}{\overline{2p+q}|^2-qq}$. If now we take this equation from twice the laſt there will remain 2 log. $\frac{p+q}{p}$ − log. $\frac{p+q}{p} = 2$ log. $\frac{2p+2q}{2p+q}$ + 2 log. $\frac{2p+q}{2p}$ − 2 log. $\frac{2p+2q}{2p+q}$ − log. $\frac{\overline{2p+q}|^2}{\overline{2p+q}|^2-qq}$: that is, log. $\frac{p+q}{p} = 2$ log. $\frac{2p+q}{2p}$ − log. $\frac{\overline{2p+q}|^2}{\overline{2p+q}|^2-qq}$. Q. E. D.

COROLLARY.

Putting $q=1$, and $n=p$, as above, we have

$$\log. \frac{n+1}{n} = 2 \log. \frac{2n+1}{2n} - \log. \frac{\overline{2n+1}|^2}{\overline{2n+1}|^2-1}.$$

I ſhall now ſet down ſome examples of the uſe of theſe theorems beginning with theorem 1.

The firſt example of the utility of this theorem may be in computing the logarithm of the number 2.

It is well known to mathematicians, that the computation of this logarithm was formerly a very laborious

 taſk:

taſk: and although the work be much ſhortened by help of the converging ſeries invented by the illuſtrious Sir ISAAC NEWTON, ſtill the logarithm of 2 has not been directly computed without many figures by any theorem I have yet ſeen. The eaſieſt computation of it that has come to my hands is in page 44. of the late ingenious Mr. THOMAS SIMPSON's pamphlet on Trigonometry and logarithms. His ſeries conſiſts of the powers of $\frac{1}{3}$.

If now we put $n = 1$ in the theorem (log. $\frac{n+1}{n} = 2 \log. \frac{2n+2}{2n+1} + \log. \frac{\overline{2n+1}|^2}{\overline{2n+1}|^2 - 1}$) we ſhall have log. $\frac{2}{1} = 2 \log. \frac{4}{3} + \log. \frac{9}{8}$. Here then the fractions, whoſe odd powers are to be uſed, are $\frac{1}{7}$ and $\frac{1}{17}$; conſequently, in the ſeries formed from $\frac{1}{7}$, about one half of the number of terms taken by Mr. SIMPSON will give the reſult true to as many places of figures as his; and, from the fraction $\frac{1}{17}$, much fewer terms will ſuffice. To ſhow how faſt theſe ſeries converge I will ſet down of each terms, enough to give the logarithm of 2 true to ten places of figures.

The odd powers of $\frac{1}{7}$ divided by their respective indices.

1st,	0·14285714286
3d,	0·00097181730
5th,	0·00001189980
7th,	0·00000017347
9th,	0·00000000275
11th,	0·00000000004
The sum,	0·14384103622 is $\frac{1}{2}$ l. of $\frac{4}{3}$.
	4
	0·5753641488 twice l. $\frac{4}{3}$.

The odd powers of $\frac{1}{17}$ divided by their respective indices.

1st,	0·05882352941
3d,	0·00006784721
5th,	0·00000014086
7th,	0·00000000035
The sum,	0·05889151783 is $\frac{1}{2}$ l. of $\frac{9}{8}$.
	2
Log. $\frac{9}{8}$,	0·117783035 66
2 log. $\frac{4}{3}$,	0·57536414488
Log. of 2.	0·69314718054

But it is obvious, that this operation gives not only the logarithm of 2 but that of 3 alſo: for the logarithm of 4 being given from that of 2, and the logarithm of $\frac{4}{3}$ computed above, the logarithm of 3 is had, being = log. of 4 − log. of $\frac{4}{3}$.

Log. of 4	1·38629436108
Log. of $\frac{4}{3}$	0·287682072 44
Log. of 3	1·09861228864

Other examples of the uſe of theſe theorems in ſhewing how eaſily the logarithms of great fractions are derived from thoſe of ſmall ones.

If the logarithm of $\frac{64}{63}$ were given, or computed, we may very eaſily find the logarithm of $\frac{32}{31}$: for (by theorem

orem 1.) $2 \log. \frac{64}{63} + \log. \frac{\overline{63}|^2}{\overline{63}|^2 - 1} = \log. \frac{32}{31}$. Here the fraction, whose odd powers are to be used in the series, is $\frac{1}{7937}$, and the very first term of it, will give the logarithm true to twelve places of figures.

Again, if the logarithm of $\frac{16}{15}$ were to be computed from that of $\frac{32}{31}$ found above, we should have $2 \log. \frac{32}{31} + \log. \frac{\overline{31}|^2}{\overline{31}|^2 - 1} = \log. \frac{16}{15}$. Here the fraction to be used in the series is $\frac{1}{1921}$, the first term of which will give the logarithm true to ten places of figures.

In like manner, from the logarithm of $\frac{16}{15}$ we may find that of $\frac{8}{7}$; from logarithm of $\frac{8}{7}$ that of $\frac{4}{3}$; and from the logarithm of $\frac{4}{3}$ that of $\frac{2}{1}$, as is done above. The respective fractions for the series will be $\frac{1}{449}$, $\frac{1}{97}$, and $\frac{1}{17}$.

Thus far the fractions I have taken have even numbers for their numerators; let us now take one whose numerator is an odd number $\frac{9}{8}$. Here n being $= 3\frac{1}{2}$, $\log. \frac{9}{7} \left(\frac{4\frac{1}{2}}{3\frac{1}{2}}\right) = 2 \log. \frac{9}{8} + \log. \frac{64}{63}$; and the fraction whose odd powers are to be used is $\frac{1}{127}$. Hence we have the log. of $\frac{8}{7}$ (for $\frac{9}{7} \div \frac{9}{8} = \frac{8}{7}$) and may proceed to find the logarithm of 2 as above. But the logarithm of $\frac{8}{7}$ may be directly derived from the equation thus: the equation in other terms is, $\log. 9 - \log. 7 = 2 \log. 9 - 2 \log. 8 + \log. \frac{64}{63}$; then, by transposition, $\log. 8 - \log. 7 = \log. 9 - \log. 8 + \log. \frac{64}{63}$, or $\log. \frac{8}{7} = \log. \frac{9}{8} + \log. \frac{64}{63}$.

 But

But when the numerator of the fraction, whose logarithm is given, is odd, theorem 2. is more commodious. For taking $\frac{9}{8}$, as before, we have $2 \log. \frac{9}{8} - \log. \frac{81}{80} = \log. \frac{5}{4}$, where the fraction to be involved is $\frac{1}{161}$. Again, $2 \log. \frac{5}{4} - \log. \frac{25}{24} = \log. \frac{3}{2}$, where the fraction is $\frac{1}{49}$. And $2 \log. \frac{3}{2} - \log. \frac{9}{8} = \log. \frac{2}{1}$, where we have only to take the difference of logarithms, as the logarithm of $\frac{9}{8}$ as well as that of $\frac{3}{2}$ is given.

All the above calculations are of hyperbolic logarithms; but the same theorems hold good for Mr. BRIGGS's, or any other. I will give an example in the computation of BRIGGS's logarithm of 7 from others already known.

Let the logarithms of 100, 99, and 50, be given; then (by theorem 1.) $2 \log. \frac{100}{99} + \frac{\overline{99}|^2}{\overline{99}|^2 - 1} = \log. \frac{50}{49}$, or $\log. \frac{100}{99} + \frac{1}{2} \log. \frac{\overline{99}|^2}{\overline{99}|^2 - 1} = \frac{1}{2} \log. \frac{50}{49}$; and then $\frac{1}{2} \log. 50 - \frac{1}{2} \log. \frac{50}{49} = \frac{1}{2} \log. 49 = \log. 7$.

Log. of $\frac{100}{99}$	- - -	0·0043648054024 5
$\frac{1}{2}$ log. of $\frac{\overline{99}|^2}{\overline{99}|^2 - 1}$	$\left(= \frac{0{\cdot}43429, \&c.}{19601} = \right)$	0·0000221567512 8
$\frac{1}{2}$ log. of $\frac{50}{49}$	- - -	0·0043869621537 3
$\frac{1}{2}$ log. of 50	- -	0·8494850021680 1
Log. of 7	- -	0·8450980400142 8

SCHOLIUM.

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Neither the number 2, nor the fraction $\frac{64}{63}$ is chosen as the most advantageous to begin with in computing a table of logarithms; but they are taken as some of the first that occurred, to show the use of these theorems. Perhaps there are other instances in which they would be shown to much more advantage; but I hope their use will appear from the few examples given. They may indeed be transformed so as to be more commodious in particular cases, and there may be some others derived from them, one or two of which I will here put down.

It is evident from theorem 1. and 2. that 2 log. $\frac{2p+2q}{2p+q}$ + log. $\frac{\overline{2p+q}|^2}{\overline{2p+q}|^2-qq}$ = 2 log. $\frac{2p+q}{2p}$ − log. $\frac{\overline{2p+q}|^2}{\overline{2p+q}|^2-qq}$; consequently, 2 log. $\frac{2p+2q}{2p+q}$ + 2 log. $\frac{\overline{2p+q}|^2}{\overline{2p+q}|^2-qq}$ = 2 log. $\frac{2p+q}{2p}$, or, log. $\frac{2p+q}{2p}$ = log. $\frac{2p+2q}{2p+q}$ + log. $\frac{\overline{2p+q}|^2}{\overline{2p+q}|^2-qq}$, which may be called theorem 3.

Again, this equation may be thus expressed: log. $\overline{2p+q}$ − log. $2p$ = log. $\overline{2p+2q}$ − log. $\overline{2p+q}$ + log. $\frac{\overline{2p+q}|^2}{\overline{2p+q}|^2-qq}$; and, by transposition, 2 log. $\overline{2p+q}$ = log. $\overline{2p+2q}$ + log. $2p$ + log. $\frac{\overline{2p+q}|^2}{\overline{2p+q}|^2-qq}$, which may be called theorem 4. And this is, in effect, one of the theorems given by Dr. HALLEY,

LEY, in the Philoſophical Tranſactions, N° 216, and of which the doctor ſaid, it converges ſo very faſt, that, in his opinion, nothing better was to be hoped.

There are yet ſome contrivances different from thoſe mentioned in the beginning of SHERWIN'S book of mathematical tables, or any other that have come to my hands, whereby the labour of computing a table of logarithms is ſhortened; but to explain them would require more time than my preſent ſituation affords me.

The obſervations and reaſonings which led me to the diſcovery of the above theorems, I imagine, need not here be mentioned. Such as they are, I beg leave to lay them before the candid and ſkilful in theſe matters, in hopes that the invention will appear to them, as it does to me, a uſeful one.

It has, indeed, been objected, by a gentleman of my acquaintance, that improvements in the conſtruction of logarithms cannot now be uſeful, becauſe logarithms are already conſtructed.

I anſwer, that argument, if it has any weight, operates equally againſt Sir ISAAC NEWTON, Dr. HALLEY, Mr. COTES, Mr. SIMPSON, and ſeveral other ingenious mathematicians; for logarithms were invented, and tables of them conſtructed, before their time; ſo that if I ſhould be thought to have miſemployed my time in

improving

improving the computation of theſe artificial numbers, I have ſome conſolation in thinking that I have therein followed the example of the very reſpectable company juſt mentioned.

I truſt, however, that, with mathematicians, every improvement in calculation will be acceptable.